Glueball filled with the quark field as a model of nucleon

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Abstract

On the basis of the non-perturbative Heisenberg's quantization scheme and using some simplifications and assumptions the reduction from the gluon-quark Lagrangian to a scalar-fermion Lagrangian is made. The corresponding field equations have a regular spherically symmetric solution which can be interpreted as a simplified model of nucleon. Some properties of such a "nucleon" are discussed: the presence of a mass gap and spin $\neq \hbar/2$. The mass gap describes the difference between the "nucleon" and the "nucleon" without quark field (a scalar glueball). The presence of spin $\neq \hbar/2$ can be connected with the spin problem of nucleon.

1 Introduction

One can assume that removing the quarks from a nucleon gives us a glueball. In this case one of the methods to prove a glueball model is by filling up it with fermions (quarks) and comparing it with nucleon properties. In Ref. [1] a scalar model of glueball is presented. This model is based on a non-perturbative quantization technique which was applied by Heisenberg for a non-linear spinor field. In our approach we have applied this idea for quantum chromodynamics. The essence of this non-perturbative technique is that in the first approximation the field degrees of freedom (A^B_μ gauge potential, $B = 1, 2, \dots 8$ is the color indices, $\mu = 0, 1, 2, 3$ is the world indices) can be reduced to scalar fields ϕ^B. The components $\phi^a(a = 1, 2, 3)$ and ϕ^m $(m = 4, 5, 6, 7, 8)$ of the scalar field have a different dynamical behavior. The numerical analysis shows that the corresponding field equations for $\phi^{a,m}$ have regular solutions only for some values of parameters of the model. The solution looks like a bag, into which we would like to put quarks (fermions).

2 Short description of the model

In this section we would like to derive the field equations describing in an approximate manner quantized fermion and SU(3) gauge fields. Strictly speaking the

Heisenberg's quantization procedure [2] is based on an infinite set of equations relating all Green's functions which are similar to Dayson-Schwinger equations but on the non-perturbative language. Of course such a set of equations is mathematically extremely hard for solving and we need for some physical reasonings for a closure. In the approach presented here we consider only 2 and 4-point Green's functions. In this case we can average the quantum chromodynamics Lagrangian and derive equations describing 2 and 4-point Green's functions. For this purpose we assume that a 4-point Green's function can be presented as some sum of the products of 2-point Green's functions, and 2-point Green's function can be decomposed as the product of scalar fields. The Lagrangian of the SU(3) gauge field interacting with quarks is

$$\widehat{\mathcal{L}} = \widehat{\mathcal{L}}_g + \widehat{\mathcal{L}}_q = -\frac{1}{4}\widehat{F}^A_{\mu\nu}\widehat{F}^{A\mu\nu} + i\widehat{\bar{\psi}}\gamma^\mu\left(\partial_\mu + iG\widehat{A}_\mu\right)\widehat{\psi} - M\widehat{\bar{\psi}}\psi \tag{1}$$

where $\widehat{\mathcal{L}}_{g,q}$ are the gluon and quark Lagrangians correspondingly; $\widehat{F}^B_{\mu\nu} = \partial_\mu \widehat{A}^B_\nu - \partial_\nu \widehat{A}^B_\mu + G f^{BCD}\widehat{A}^C_\mu \widehat{A}^D_\nu$ is the field strength operator; $B, C, D = 1, \ldots, 8$ are the SU(3) color indices; G is the coupling constant; f^{BCD} are the structure constants for the SU(3) gauge group; $\widehat{A}^B_\mu$ is the gauge potential operator; $\widehat{A}_\mu = L^B \widehat{A}^B_\mu$; L^B are the generators of the SU(3) gauge group; $\widehat{\psi}$ is the quark field and M is the mass matrix of quarks. In order to derive equations describing the quantized field we average this Lagrangian over a quantum state $\left|Q\right\rangle$

$$\left\langle Q \left| \widehat{\mathcal{L}}(x) \right| Q \right\rangle = \left\langle \widehat{\mathcal{L}} \right\rangle = \left\langle \widehat{\mathcal{L}}_g \right\rangle + \left\langle \widehat{\mathcal{L}}_q \right\rangle \tag{2}$$

where

$$\begin{aligned}
\left\langle \widehat{\mathcal{L}}_g \right\rangle &= -\Bigg[\frac{1}{2}\left\langle \left(\partial_\mu \widehat{A}^B_\nu(x)\right)\left(\partial^\mu \widehat{A}^{B\nu}(x)\right) - \left(\partial_\mu \widehat{A}^B_\nu(x)\right)\left(\partial^\nu \widehat{A}^{B\mu}(x)\right)\right\rangle \\
&= \frac{1}{2} G f^{BCD}\left\langle \left(\partial_\mu \widehat{A}^B_\nu(x) - \partial_\nu \widehat{A}^B_\mu(x)\right)\widehat{A}^{C\mu}(x)\widehat{A}^{D\nu}(x)\right\rangle \\
&+ \frac{1}{4} G^2 f^{BC_1D_1} f^{BC_2D_2}\left\langle \widehat{A}^{C_1}_\mu(x)\widehat{A}^{D_1}_\nu(x)\widehat{A}^{C_2\mu}(x)\widehat{A}^{D_2\nu}(x)\right\rangle\Bigg], \qquad (3) \\
\left\langle \widehat{\mathcal{L}}_q \right\rangle &= i\widehat{\bar{\psi}}\gamma^\mu\left(\partial_\mu + iG\widehat{A}_\mu\right)\widehat{\psi} \qquad (4)
\end{aligned}$$

At first we will calculate $\left\langle \widehat{\mathcal{L}}_g \right\rangle$ following Refs. [1, 3]. One can see that schematically we have the following 2, 3, and 4-point Green's functions: $\left\langle (\partial A)^2 \right\rangle$, $\left\langle (\partial A)\, A^2 \right\rangle$ and $\left\langle (A)^4 \right\rangle$. We suppose that the odd Green's functions can be written as the following product:

$$\left\langle \widehat{A}^B_\alpha(x)\widehat{A}^C_\beta(y)\widehat{A}^D_\gamma(z)\right\rangle \approx \left\langle \widehat{A}^B_\alpha(x)\widehat{A}^C_\beta(y)\right\rangle \left\langle \widehat{A}^D_\gamma(z)\right\rangle + (\text{other permutations}) = 0 \tag{5}$$

as we suppose that $\langle \widehat{A}^B_\alpha(x)\rangle = 0$. Further we suppose that a 2-point Green's function can be presented in the so-called one-function approximation [3] as

$$\left\langle \widehat{A}^B_\alpha(x)\widehat{A}^C_\beta(y)\right\rangle = \mathcal{G}^{BC}_{\alpha\beta}(x,y) \approx -\eta_{\alpha\beta} f^{BAD} f^{CAE}\phi^D(x)\phi^E(y) \tag{6}$$

where $\phi^A(x)$ is the scalar field which describes the 2-point Green's function. These two assumptions are similar to the quantum harmonic oscillator where $\langle x\rangle = 0$ but $\left\langle x^2\right\rangle \neq 0$. The 4-point Green's function can be written in a one-function approximation as in the symmetrized product of corresponding two 2-point Green's functions

$$\begin{aligned}&\left\langle \widehat{A}^B_\alpha(x)\widehat{A}^C_\beta(y)\widehat{A}^D_\gamma(z)\widehat{A}^R_\delta(u)\right\rangle \approx \left\langle \widehat{A}^B_\alpha(x)\widehat{A}^C_\beta(y)\right\rangle \left\langle \widehat{A}^D_\gamma(z)\widehat{A}^R_\delta(u)\right\rangle \\ &+ \left\langle \widehat{A}^B_\alpha(x)\widehat{A}^D_\gamma(z)\right\rangle \left\langle \widehat{A}^C_\beta(y)\widehat{A}^R_\delta(u)\right\rangle + \left\langle \widehat{A}^B_\alpha(x)\widehat{A}^R_\delta(u)\right\rangle \left\langle \widehat{A}^C_\beta(y)\widehat{A}^D_\gamma(z)\right\rangle .\end{aligned} \tag{7}$$

Taking into account these expressions for the 2, 3, and 4-point Green's functions we can derive the effective Lagrangian $\mathcal{L}_\phi = \left\langle \widehat{\mathcal{L}}_g\right\rangle$ for the scalar field ϕ^A which describes 2 and 4-point Green's functions

$$\begin{aligned}\mathcal{L}_\phi = \langle \mathcal{L}_g\rangle \approx \frac{4}{G^2}\Bigg[&\frac{1}{2}\left(\partial_\mu\phi^A\right)^2 - \frac{\lambda_2}{4}\left[\phi^a\phi^a - \phi^a_0\phi^a_0\right]^2 + \frac{\lambda_2}{4}\left(\phi^a_0\phi^a_0\right)^2 \\ &- \frac{\lambda_1}{4}\left[\phi^m\phi^m - \phi^m_0\phi^m_0\right]^2 + \frac{\lambda_2}{4}\left(\phi^m_0\phi^m_0\right)^2 - \left(\phi^a\phi^a\right)\left(\phi^m\phi^m\right)\Bigg]\end{aligned} \tag{8}$$

where the indices $a = 1,2,3$ are SU(2) indices and $m = 4,5,6,7,8$ are the coset SU(3)/SU(2) indices; ϕ^A_0 are some constants. We can add the term $-\lambda_2(\phi^m_0\phi^m_0)^2/4$, which is inessential for the dynamics (but essential for the finiteness of the energy), and the Lagrangian becomes

$$\begin{aligned}\mathcal{L}_\phi = \langle \mathcal{L}_g\rangle = \frac{4}{G^2}\Bigg[&\frac{1}{2}\left(\partial_\mu\phi^A\right)^2 - \frac{\lambda_2}{4}\left[\phi^a\phi^a - \phi^a_0\phi^a_0\right]^2 \\ &- \frac{\lambda_1}{4}\phi^m\phi^m\left[\phi^m\phi^m - 2\phi^m_0\phi^m_0\right]^2 - \left(\phi^a\phi^a\right)\left(\phi^m\phi^m\right)\Bigg].\end{aligned} \tag{9}$$

The next step is calculation of $\langle \mathcal{L}_q\rangle$,

$$\left\langle \widehat{\mathcal{L}}_q\right\rangle = \left\langle \widehat{\bar{\psi}}\Big(i\gamma^\mu\partial_\mu - M\Big)\widehat{\psi}\right\rangle - G\left\langle \widehat{\bar{\psi}}\gamma^\mu\widehat{A}_\mu\widehat{\psi}\right\rangle . \tag{10}$$

We see that the fermion (quark) field ψ does not have any strong self-interaction. Therefore we can suppose that in the first approximation the dynamical behavior of the quantized quark field ψ is similar to the dynamics of classical field

$$\left\langle \widehat{\mathcal{L}}_q\right\rangle \approx \bar{\psi}\Big(i\gamma^\mu\partial_\mu - M\Big)\psi - G\bar{\psi}\gamma^\mu\left\langle \widehat{A}_\mu\right\rangle\psi. \tag{11}$$

Here, we have to note that according to our previous assumptions $\left\langle A^B_\mu\right\rangle = 0$ and consequently the second term in Eq. (11) is zero but the interaction between

gluon and quark fields *must* exist. To introduce the interaction between ϕ^A and ψ we have to do some physical assumptions about this interaction and insert this term into Lagrangian by hand. For example, one can assume that the quark field *interacts with fluctuations of the gluon field*,

$$\mathcal{L}_{int} = -\alpha' \left(\bar{\psi}\psi\right)\left(\widehat{A}^B_\mu \widehat{A}^{B\mu}\right) = \alpha\left(\bar{\psi}\psi\right)\left(\phi^A\phi^A\right) \tag{12}$$

where for the redefinition of α' we have used Eq. (6). This assumption is equivalent to cutting off the infinite set of equations connecting all Green's functions for exactly quantized gluon and quark fields. Thus the quark Lagrangian is

$$\left\langle \widehat{\mathcal{L}}_q \right\rangle \approx \bar{\psi}\left[i\gamma^\mu\partial_\mu - \left(M - \alpha\phi^A\phi^A\right)\right]\psi. \tag{13}$$

Hereafter, we omit flavor indices to simplify notation. Finally we have the following effective Lagrangian describing the non-perturbative quantized qluon and quark fields interacting amongst themselves

$$\begin{aligned}\left\langle \widehat{\mathcal{L}} \right\rangle = \frac{4}{G^2}\Bigg[\frac{1}{2}\left(\partial_\mu\phi^A\right)^2 - \frac{\lambda_2}{4}\left[\phi^a\phi^a - \phi^a_0\phi^a_0\right] \\ -\frac{\lambda_1}{4}\phi^m\phi^m\left[\phi^m\phi^m - 2\phi^m_0\phi^m_0\right] - \left(\phi^a\phi^a\right)\left(\phi^m\phi^m\right)\Bigg] \\ +\bar{\psi}\left[i\gamma^\mu\partial_\mu - \left(M - \alpha\phi^A\phi^A\right)\right]\psi\end{aligned} \tag{14}$$

with the following field equations

$$\nabla^\mu\nabla_\mu\phi^a = -\phi^a\left[2\phi^m\phi^m + \lambda_2\left(\phi^a\phi^a - \phi^a_0\phi^a_0\right) - \frac{\alpha G^2}{2}\bar{\psi}\psi\right], \tag{15}$$

$$\nabla^\mu\nabla_\mu\phi^m = -\phi^m\left[2\phi^a\phi^a + \lambda_1\left(\phi^m\phi^m - \phi^m_0\phi^m_0\right) - \frac{\alpha G^2}{2}\bar{\psi}\psi\right], \tag{16}$$

$$\left[i\gamma^\mu\partial_\mu - \left(M - \alpha\phi^A\phi^A\right)\right]\psi = 0. \tag{17}$$

3 Initial equations

In this section we would like to present the solution which later we will view as a simplified model of the nucleon. We search solution in the following spherically symmetric form:

$$\phi^a(r) = \frac{\phi(r)}{\sqrt{6}}, \quad a = 1,2,3; \tag{18}$$

$$\phi^m(r) = \frac{f(r)}{\sqrt{10}}, \quad m = 4,5,6,7,8; \tag{19}$$

$$\psi\left(t,r,\theta,\varphi\right) = \mathrm{e}^{-iEt}\begin{pmatrix} h(r) \\ 0 \\ g(r)\cos\theta \\ g(r)\sin\theta\mathrm{e}^{-i\varphi}\end{pmatrix} \tag{20}$$

where r, θ, φ are spherical coordinates. Once again we would like to remind that in our approach we suppose that the components of scalar fields with different indices $\phi^a, a = 1, 2, 3$ and $\phi^m, m = 4, 5, 6, 7$ have different dynamical behavior. After s ubstituting Eqs. (18)-(20) into Eqs. (15)-(17) gives

$$
\begin{aligned}
\phi'' + \frac{2}{r}\phi' &= \phi\left[f^2 + \lambda_2\left(\phi^2 - m^2\right) - \frac{\alpha G^2}{2}\left(h^2 - g^2\right)\right], && (21)\\
f'' + \frac{2}{r}f' &= f\left[\phi^2 + \lambda_1\left(f^2 - \mu^2\right) - \frac{\alpha G^2}{2}\left(h^2 - g^2\right)\right], && (22)\\
h' + \frac{2}{r}h &= g\left[E + M - \frac{\alpha}{2}\left(f^2 + \phi^2\right)\right], && (23)\\
g' &= -h\left[E - M + \frac{\alpha}{2}\left(f^2 + \phi^2\right)\right] && (24)
\end{aligned}
$$

where $2\phi^a_0\phi^a_0 = m^2$ and $2\phi^m_0\phi^m_0 = \mu^2$; m, μ are some constants which will be calculated by solving Eqs. (21)-(24); and constants $\lambda_{1,2}$ are redefined $\lambda_{1,2}/2 \to \lambda_{1,2}$. We redefine $\phi(r)/\phi(0) \to \phi(x)$, $f(r)/\phi(0) \to f(x)$, $m/\phi(0) \to m$, $\mu/\phi(0) \to \mu$, $h(r)/\phi^{3/2}(0) \to h(x)$, $g(r)/\phi^{3/2}(0) \to g(x)$, $\alpha/\phi(0) \to \alpha$, $E/\phi(0) \to E$, $M/\phi(0) \to M$ and introduce the dimensionless radius $x = r\phi(0)$. After this we have the following set of equations:

$$
\begin{aligned}
\phi'' + \frac{2}{x}\phi' &= \phi\left[f^2 + \lambda_2\left(\phi^2 - m^2\right) - \frac{\alpha G^2}{2}\left(h^2 - g^2\right)\right], && (25)\\
f'' + \frac{2}{x}f' &= f\left[\phi^2 + \lambda_1\left(f^2 - \mu^2\right) - \frac{\alpha G^2}{2}\left(h^2 - g^2\right)\right], && (26)\\
h' + \frac{2}{x}h &= g\left[E + M - \frac{\alpha}{2}\left(f^2 + \phi^2\right)\right], && (27)\\
g' &= -h\left[E - M + \frac{\alpha}{2}\left(f^2 + \phi^2\right)\right] && (28)
\end{aligned}
$$

where $d(\cdots)/dx = (\cdots)'$. Evidently these equations are too complicated to be solved analytically. Preliminary numerical investigations show us that the set (25)-(28) does not have regular solutions for arbitrary choice of the parameters m, μ, and E. We will solve equations (25)-(28) as a nonlinear eigenvalue problem for eigenstates $\phi(x), f(x), h(x), g(x)$ and eigenvalues m, μ, E, i.e. we have to find parameters m, μ, E to provide existence of regular functions $\phi(r)$, $f(r)$, $h(r)$ and $g(r)$. We will search for the regular solution under the following boundary conditions

$$
\begin{aligned}
\phi(0) &= 1, \quad \phi(\infty) = m; && (29)\\
f(0) &= f_0, \quad f(\infty) = 0; && (30)\\
h(0) &= 0; && (31)\\
g(0) &= g_0. && (32)
\end{aligned}
$$

The boundary conditions for $\phi(x)$ and $f(x)$ allow us to say that these functions resemble interacting kink and soliton.

For the regular solution which will be presented below we assign the following values: $\phi(0) = 1.0; f(0) = \sqrt{0.6}; g(0) = 0.1; G = 2.0; M = 0.2; \lambda_1 = 1.0; \lambda_2 = 0.1; \alpha = 1.0$.

4 Numerical solution

In accordance Ref. [1], we use the following numerical method for solving Eqs. (25)-(28): we take a zero approximation for the functions $f(x), h(x), g(x)$ (which are $f_0(x) = 0.6/\cosh^2(x/4)$ and $h_0(x) = g_0(x) = 0$) and solve equation (25) in the following form:

$$\phi_1'' + \frac{2}{x}\phi_1' = \phi_1\left[f_0^2 + \lambda_2\left(\phi_1^2 - m_1^2\right)\right] \tag{33}$$

where m_1 is the first approximation for the parameter m, the boundary conditions are (29) and the function $\phi_1(x)$ is the first approximation for the function $\phi(x)$; $\phi_1(x)$ and m_1 are the eigenfunction and the eigenvalue for Eq. (25) correspondingly, with the boundary conditions (29). Obtaining the regular solution $\phi_1(x)$ we can substitute it into Eq. (26) with zero approximations $h_0(x) = g_0(x) = 0$ and solve the equation

$$f_1'' + \frac{2}{x}f_1' = f_1\left[\phi_1^2 + \lambda_1\left(f_1^2 - \mu_1^2\right)\right] \tag{34}$$

with the boundary conditions (30). $f_1(x)$ and μ_1 are correspondingly the eigenfunction and the eigenvalue for Eq.(26), with the boundary conditions (30).

Further having the first approximations $\phi_1(x), f_1(x)$ we can substitute them into Eqs. (27) and (28)

$$h_1' + \frac{2}{x}h_1' = g_1\left[E + M - \frac{\alpha}{2}\left(f_1^2 + \phi_1^2\right)\right], \tag{35}$$

$$g_1' = -h_1\left[E - M + \frac{\alpha}{2}\left(f_1^2 + \phi_1^2\right)\right] \tag{36}$$

where $h_1(x), g_1(x), E_1$ are eigenfunctions and eigenvalue for Dirac equations (35) and (36) with the potential $\frac{\alpha}{2}(f_1^2 + \phi_1^2)$. Thus having the first approximations for $\phi_1(x), f_1(x), h_1(x), g_1(x), m_1, \mu_1, E_1$ one can substitute these quantities into Eq. (25),

$$\phi_2'' + \frac{2}{x}\phi_2' = \phi_2\left[f_1^2 + \lambda_2\left(\phi_2^2 - m_1^2\right) - \frac{\alpha G^2}{2}\left(h_1^2 - g_1^2\right)\right]. \tag{37}$$

This equation gives us the second approximation $\phi_2(x)$ for the function $\phi(x)$,

and so on. Thus, on the i^{th} step we have

$$\phi_i'' + \frac{2}{x}\phi_i' = \phi_i \left[f_{i-1}^2 + \lambda_1 \left(\phi_i^2 - m_i^2 \right) - \frac{\alpha G^2}{2} \left(h_{i-1}^2 - g_{i-1}^2 \right) \right] \tag{38}$$

$$f_i'' + \frac{2}{x}f_i' = f_i \left[\phi_i^2 + \lambda_1 \left(f_i^2 - \mu_i^2 \right) - \frac{\alpha G^2}{2} \left(h_{i-1}^2 - g_{i-1}^2 \right) \right], \tag{39}$$

$$h_i' + \frac{2}{x}h_i' = g_i \left[E_i + M - \frac{\alpha}{2} \left(f_i^2 + \phi_i^2 \right) \right], \tag{40}$$

$$g_i' = -h_i \left[E_i - M + \frac{\alpha}{2} \left(f_i^2 + \phi_i^2 \right) \right] \tag{41}$$

where the functions $f_{i-1}(x), h_{i-1}(x), g_{i-1}(x)$ are defined in the preceding step,

$$f_{i-1}'' + \frac{2}{x}f_{i-1}' = f_{i-1} \left[\phi_{i-1}^2 + \lambda_1 \left(f_{i-1}^2 - \mu_{i-1}^2 \right) - \frac{\alpha G^2}{2} \left(h_{i-2}^2 - g_{i-2}^2 \right) \right], \tag{42}$$

$$h_{i-1}' + \frac{2}{x}h_{i-1}' = g_{i-1} \left[E_{i-1} + M - \frac{\alpha}{2} \left(f_{i-1}^2 + \phi_{i-1}^2 \right) \right], \tag{43}$$

$$g_{i-1}' = -h_{i-1} \left[E_{i-1} - M + \frac{\alpha}{2} \left(f_{i-1}^2 + \phi_{i-1}^2 \right) \right] \tag{44}$$

Such procedure has to lead to precise eigenfunctions

$$\phi_i^*(x) \to \phi^*(x), \quad f_i^*(x) \to f^*(x), \quad h_i^*(x) \to h^*(x), \quad g_i^*(x) \to g^*(x) \tag{45}$$

with precise eigenvalues

$$m_i^* \to m^*, \quad \mu_i^* \to m^*, \quad E_i^* \to E^* \tag{46}$$

where $\phi_i^*, f_i^*, h_i^*, g_i^*$ are the regular solutions of Eqs. (38)-(41) which can be regular only for the values of parameters m_i^*, μ_i^*, E_i^*. For the other values of parameters $m_i \neq m_i^*, \mu_i \neq \mu_i^*, E_i \neq E_i^*$ the functions $\phi_i \neq \phi_i^*, f_i \neq f_i^*, h_i \neq h_i^*, g_i \neq g_i^*$ are singular and imply infinite energy.

4.1 More detailed description of the numerical calculations for every step

In this section we would like to describe more carefully the procedure of the numerical solution of each equation of the system (38)-(44).

At first we would like to describe more carefully the numerical solution of equation (33). For this purpose, we choose the zero approximation $f_0(x)$ as follows:

$$f_0(x) = \frac{\sqrt{0.6}}{\cosh^2 \frac{x}{4}}. \tag{47}$$

Typical solutions for the arbitrary value of m_1 are presented in Fig. 1. We see that when $m_1 < m_1^*$ (m_1^* is an eigenvalue which gives us an eigenfunction $\phi_1^*(x)$) the solution $\phi_1(x)$ is singular and closer to the singularity the equation has the form

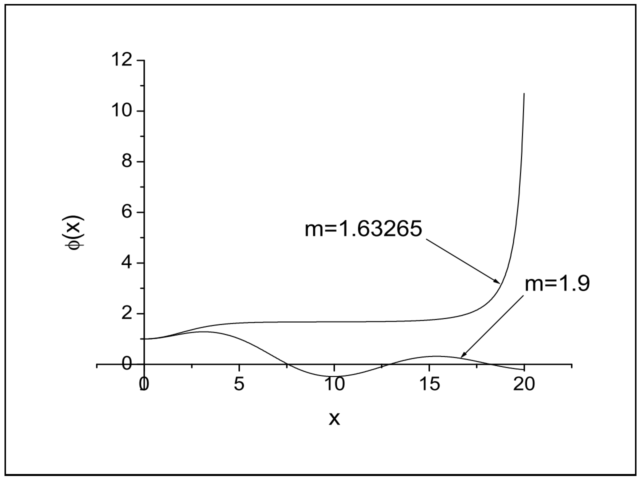

Figure 1: The typical singular solutions $\phi(x)$ of Eq. (38). The solution is presented for $i = 4$ step.

$$\phi_1'' \approx \lambda_2 \phi_1^3 \tag{48}$$

Consequently the solution is

$$\phi_1(x) \approx \sqrt{\frac{2}{\lambda_2}} \frac{1}{x_0 - x} \tag{49}$$

where x_0 is some constant depending on m_1. On the other hand, for $m_1 > m_1^*$ the solution is also presented in Fig. 1 and the corresponding asymptotical equation is

$$\phi_1''(x) + \frac{2}{x}\phi_1' \approx -\left(\lambda_2 m^2\right)\phi_1 \tag{50}$$

which has the following solution:

$$\phi_1(x) \approx \phi_\infty \frac{\sin\left(x\sqrt{\lambda_2 m^2} + \alpha\right)}{x} \tag{51}$$

where ϕ_∞ and α are some constants. This allows us to assert that there is an eigenvalue m_1^* for which an eigenfunction exists. This is presented in Fig. 5, to some degree of accuracy.

For the value m_1^* equation (33) has the following asymptotical behavior:

$$\phi_1''(x) + \frac{2}{x}\phi_1' \approx 2\lambda_2 \left(m_1^*\right)^2 \left(\phi_1 - m_1^*\right) \tag{52}$$

and the corresponding asymptotical solution is

$$\phi_1(x) \approx m_1^* + \beta_1 \frac{e^{-x\sqrt{2\lambda_2\left(m_1^*\right)^2}}}{x} \tag{53}$$

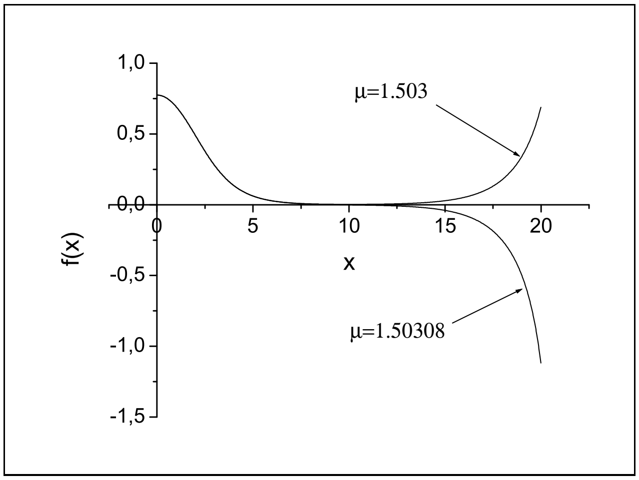

Figure 2: The typical singular solutions $f(x)$ of Eq. (39). The solution is presented for $i = 4$ step.

where β_1 is some constant.

The next step is finding the first approximation for the $f_1(x)$ function. The corresponding equation is

$$f_1'' + \frac{2}{x} f' = f_1 \left[\phi_1^2 + \lambda_1 \left(f_1^2 - \mu_1^2\right)\right]. \tag{54}$$

The numerical investigation shows that for arbitrary μ there are two different singular solutions which are presented in Fig. 2. Analogously, the singular behavior of the solution $f_1(x)$ is

$$f_1(x) \approx \sqrt{\frac{2}{\lambda_1}} \frac{1}{x - x_0} \quad \text{by} \quad \mu_1 < \mu_1^*, \tag{55}$$

$$f_1(x) \approx -\sqrt{\frac{2}{\lambda_1}} \frac{1}{x - x_0} \quad \text{by} \quad \mu_1 > \mu_1^*. \tag{56}$$

Evidently, we can suppose that there is the eigenfunction $f_1^*(x)$ for the eigenvalue $\mu_1 = \mu_1^*$ with the following asymptotical behavior:

$$f_1^*(x) \approx f_{\infty,1} \frac{e^{-x\sqrt{\left(m_0^*\right)^2 - \lambda_2 \left(\mu_1^*\right)^2}}}{x} \tag{57}$$

where $f_{\infty,1}$ is some parameter.

The next step is the substitution of the first approximations $\phi_1^*(x), f_1^*(x)$ into Eqs. (40) and (41), to find eigenfunctions $h_1^*(x), g_1^*(x)$ corresponding to the

eigenvalue E_1^*,

$$h_1' + \frac{2}{x}h = g_1 \left[E_1 + M - \frac{\alpha}{2}\left[(f_1^*)^2 + (\phi_1^*)^2\right]\right], \quad (58)$$

$$g_1 = h_1 \left[-E_1 + M - \frac{\alpha}{2}\left[(f_1^*)^2 + (\phi_1^*)^2\right]\right]. \quad (59)$$

The numerical solutions of these equations are the same as that of the Dirac equation for a quantum particle with spin in a potential hole. In our case the hole is formed by the gluon fields: $V_{eff} = \alpha(\phi^2 + f^2)/2$. The typical singular solutions of these equations are presented in Figs. 3 and 4.

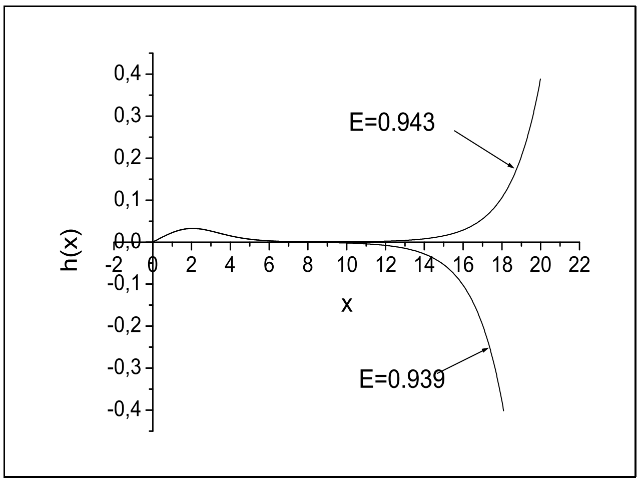

Figure 3: The typical singular solutions $h(x)$ of Eqs. (40) and (41).

Further, the eigenfunctions $f_1^*(x), h_1^*(x), g_1^*(x)$ will be substituted into equation (38) to find the eigenfunction $\phi_2^*(x)$ which corresponds to the eigenvalue $m_2 = m_2^*$, and so on.

The result of these calculations is presented in Figs. 5, 6, 7, 8 and Table 1. One can see that there is the convergence: $\phi_i^*(x) \to \phi^*(x)$, $f_i^*(x) \to f^*(x)$, $h_i^*(x) \to h^*(x)$, $g_i^*(x) \to g^*(x)$, $m_i^* \to m^*$, $\mu_i^* \to \mu^*$ and $E_i^* \to E^*$, where $f^*(x), \phi^*(x), h^*(x), g^*(x)$, are the eigenfunctions and m^*, μ^*, E^* are the eigenvalues of the nonlinear eigenvalue problem (25)-(28).

Finally we would like to mention that the presented solution allows us to avoid the Derrick's Theorem [5] which forbids the existence of regular solutions for scalar fields. One of the necessary conditions of this Theorem is that the solution at infinity tends to a global minimum of the potential energy but in our case the solution tends to a local minimum $\phi^* \to m^*, f^* \to 0$ (the global minimum is $\phi = m, f = \mu$).

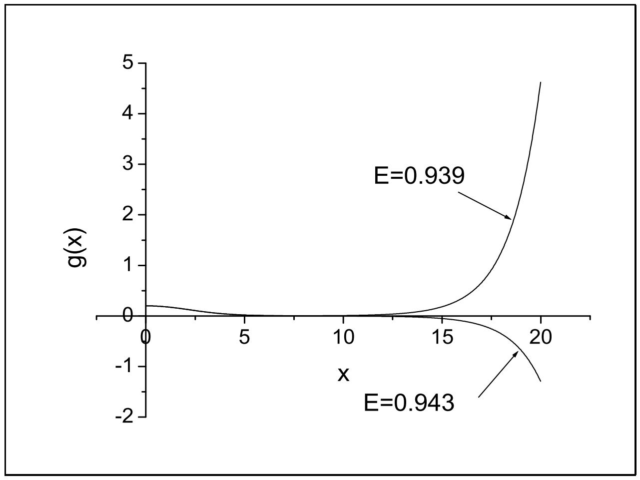

Figure 4: The typical singular solutions $g(x)$ of Eqs. (40) and (41).

i	1	2	3	4
m_i^*	1.837435109502	1.61449115	1.6296365	1.63265546
μ_i^*	1.4921052897028195	1.504847	1.5022397	1.5030292
E_i^*	0.932603482	0.942702	0.939357	0.940497

Table 1: The iterative parameters m_i^*, μ_i^* and E_i^*.

5 Properties of the solution

In this section we would like to describe properties of the derived solution. It is easy to see that the asymptotical behavior of the regular solution is

$$\phi^*(x) \approx m^* + \phi_\infty \frac{\exp\{-x\sqrt{2\lambda_2 (m^*)^2}\}}{x} \tag{60}$$

$$f^*(x) \approx f_\infty \frac{\exp\{-x\sqrt{(m^*)^2 - \lambda_1 (\mu^*)^2}\}}{x} \tag{61}$$

$$h^*(x) \approx h_\infty \frac{\exp\{-x\sqrt{\left(M - \frac{\alpha (m^*)^2}{2}\right)^2 - (E^*)^2}\}}{x} \tag{62}$$

$$g^*(x) \approx g_\infty \frac{\exp\{-x\sqrt{\left(M - \frac{\alpha (m^*)^2}{2}\right)^2 - (E^*)^2}\}}{x} \tag{63}$$

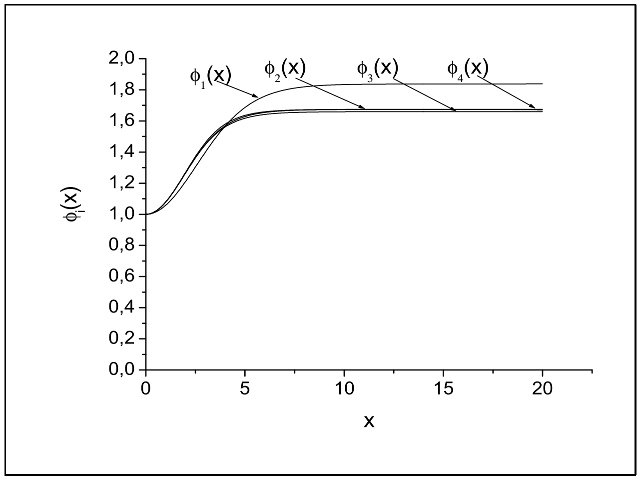

Figure 5: The iterative functions $\phi_{1,2,3,4}(x)$.

$$h_\infty = -g_\infty \sqrt{\frac{M - \alpha \left(m^*\right)^2/2 + E^*}{M - \alpha \left(m^*\right)^2/2 - E^*}} \tag{64}$$

where m^*, μ^*, E^* are the eigenvalues derived by the solving Eqs. (25)-(28).

For determination of the corresponding energy we rewrite the Lagrangian (14) in the form

$$\left\langle \widehat{\mathcal{L}} \right\rangle = \mathcal{L}_s + \mathcal{L}_{int} + \mathcal{L}_\psi \tag{65}$$

where Lagrangians for the scalar field $\mathcal{L}_s$, interaction $\mathcal{L}_{int}$, and spinor field $\mathcal{L}_\psi$ are, correspondingly,

$$\begin{aligned} \mathcal{L}_s &= \frac{4}{G^2}\left[\frac{1}{2}\left(\partial_\mu \phi^A\right)^2 - \frac{\lambda_2}{4}\left(\phi^a\phi^a - \phi^a_0\phi^a_0\right)^2 \right. \\ &\left. -\frac{\lambda_1}{4}\phi^m\phi^m\left(\phi^m\phi^m - 2\phi^m_0\phi^m_0\right) - \left(\phi^a\phi^a\right)\left(\phi^m\phi^m\right)\right], \qquad (66) \\ \mathcal{L}_{int} &= \alpha\left(\phi^A\phi^A\right)\left(\bar{\psi}\psi\right), \qquad (67) \\ \mathcal{L}_\psi &= \bar{\psi}\left(i\gamma^\mu\partial_\mu - M\right)\psi. \qquad (68) \end{aligned}$$

Then the energy density is found as follows:

$$\begin{aligned} \varepsilon(r) = &\frac{4}{G^2}\left[\frac{1}{2}\left(\partial_t\phi^A\right)^2 + \frac{1}{2}\left(\partial_i\phi^A\right)^2 + \frac{\lambda_2}{4}\left(\phi^a\phi^a - \phi^a_0\phi^a_0\right)^2 \right. \\ &\left. + \frac{\lambda_1}{4}\phi^m\phi^m\left(\phi^m\phi^m - 2\phi^m_0\phi^m_0\right) + \left(\phi^a\phi^a\right)\left(\phi^m\phi^m\right)\right] - \alpha\left(\phi^A\phi^A\right)\left(\bar{\psi}\psi\right) + \varepsilon_\psi. \end{aligned} \tag{69}$$

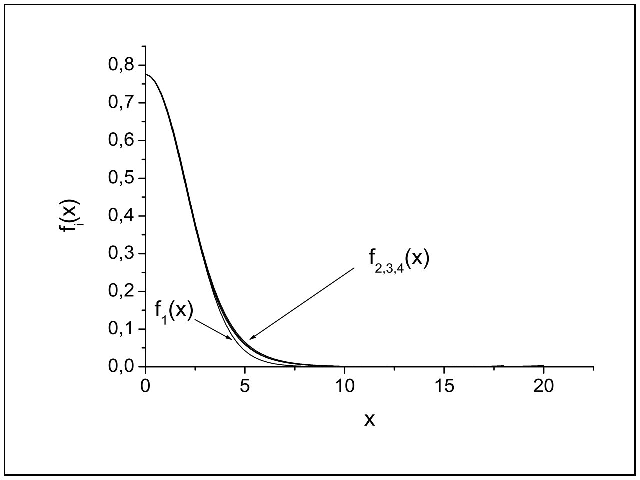

Figure 6: The iterative functions $f_{1,2,3,4}(x)$.

We will determine the energy density of the quark field ε_ψ from the energy-momentum tensor of ψ,

$$T^{\mu\nu} = \frac{i}{2} g^{\mu\alpha} \left(\bar{\psi} \gamma^\mu \frac{\partial \psi}{\partial x^\alpha} - \frac{\partial \bar{\psi}}{\partial x^\alpha} \gamma^\mu \psi \right). \tag{70}$$

Using ansätz (20) we have

$$\varepsilon_\psi = T^{00} = E \left(h^2 + g^2 \right). \tag{71}$$

Therefore

$$\varepsilon(r) = \frac{1}{G^2} \left\{ (\phi^*)'^2 + (f^*)'^2 + \frac{\lambda_2}{2} \left[(\phi^*)^2 - (m^*)^2 \right] + \frac{\lambda_1}{2} (f^*)^2 \left[(f^*)^2 - 2 (\mu^*)^2 \right] + (f^*)^2 (\phi^*)^2 \right\} - \frac{\alpha}{2} \left[(f^*)^2 + (\phi^*)^2 \right] \left[(h^*)^2 - (g^*)^2 \right] + E^* \left[(h^*)^2 + (g^*)^2 \right] \tag{72}$$

where $\lambda_{1,2}, m^*$ and μ^* are redefined according to the remark after Eq. (24), and the functions $(\cdots)$ are the eigenfunctions. Thus the energy is

$$\begin{aligned} W =& 4\pi\phi_0 \int_0^\infty x^2 \left\{ \frac{1}{G^2} \left[(\phi^*)'^2 + (f^*)'^2 + \frac{\lambda_2}{2} \left[(\phi^*)^2 - (m^*)^2 \right] \right.\right. \\ & \left. + \frac{\lambda_1}{2} (f^*)^2 \left[(f^*)^2 - 2 (\mu^*)^2 \right] + (f^*)^2 (\phi^*)^2 \right] \\ & \left. - \frac{\alpha}{2} \left[(f^*)^2 + (\phi^*)^2 \right] \left[(h^*)^2 - (g^*)^2 \right] + E^* \left[(h^*)^2 + (g^*)^2 \right] \right\} dx \\ & = 4\pi\phi_0 I_1 \left(\lambda_{1,2}, m^*, \mu^*, E^*, \alpha \right) \end{aligned} \tag{73}$$

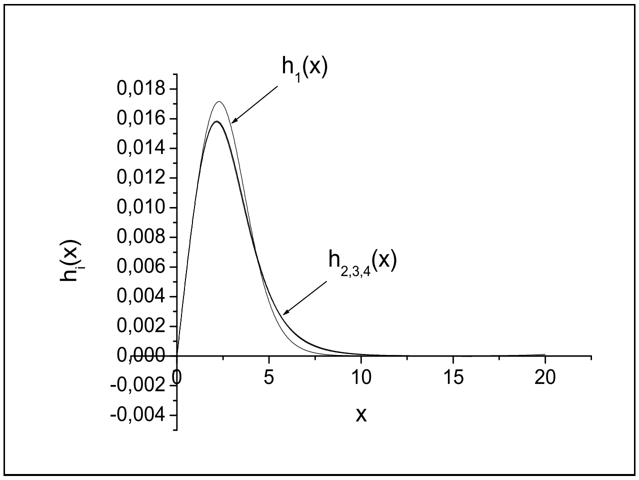

Figure 7: The iterative functions $h_{1,2,3,4}(x)$.

where the quantities $r, f^*, \phi^*, h^*, g^*, m^*, \mu^*, E^*, \alpha$ are dimensionless according to the remark after Eq. (24). The quantity $\phi(0)^{-1}$ defines the radius of the object as a consequence of the asymptotical behavior (60)-(64). The profile of the energy density is presented in Fig. 9. The numerical calculations for the integral I_1 gives

$$
\begin{aligned}
I_1 = \int\limits_0^\infty x^2 \Biggl\{ & \frac{1}{G^2}\left[(\phi^*)'^2 + (f^*)'^2 + \frac{\lambda_2}{2}\left[(\phi^*)^2 - (m^*)^2\right]\right. \\
= & \frac{\lambda_1}{2}(f^*)^2\left[(f^*)^2 - 2(\mu^*)^2\right] + (f^*)^2(\phi^*)^2\Bigg] \\
- & \frac{\alpha}{2}\left[(f^*)^2 + (\phi^*)^2\right]\left[(h^*)^2 - (g^*)^2\right] + E^*\left[(h^*)^2 + (g^*)^2\right]\Biggr\} dx \approx 0.395.
\end{aligned}
\tag{74}
$$

Another interesting property of the solution is a total angular momentum density M_i. The corresponding operator is

$$\widehat{M_i} = \widehat{L}_i + \widehat{s}_i \tag{75}$$

where $\widehat{L}_i$ is the operator of the orbital momentum density and $\widehat{s}_i$ is the spin operator density,

$$
\begin{aligned}
\widehat{L}_i &= \epsilon_{ijk} x_j \left(-i\frac{\partial}{\partial x_k}\right), && (76)\\
\widehat{s}_1 &= i\gamma^2\gamma^3, \quad \widehat{s}_2 = i\gamma^3\gamma^1, \quad \widehat{s}_3 = i\gamma^1\gamma^2 && (77)
\end{aligned}
$$

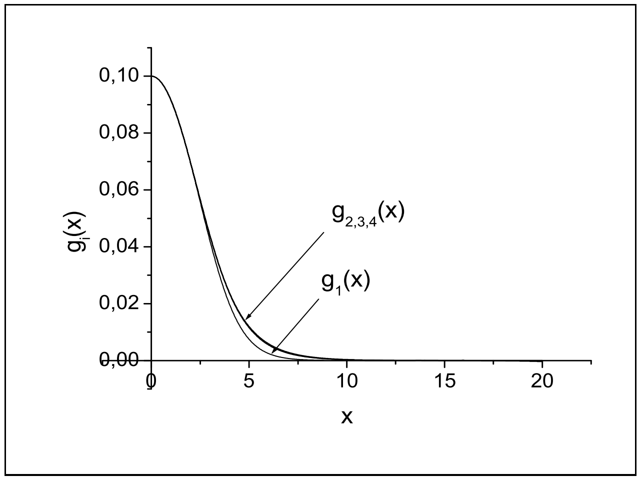

Figure 8: The iterative functions $g_{1,2,3,4}(x)$.

where γ^i denote Dirac matrices. The calculation of M_z gives us

$$\langle M_z \rangle = \frac{1}{2}\psi^* \widehat{M_z}\psi = \frac{1}{2}\left(h^2 + g^2\right), \tag{78}$$

$$\langle M_x \rangle = \langle M_y \rangle = 0. \tag{79}$$

The z-projection of the full angular momentum $\mathcal{M}_z$ is

$$\begin{aligned}\mathcal{M}_z =& \frac{1}{2}\int \psi^* \widehat{M_z}\psi dV = \frac{1}{2}\int \left(h^2 + g^2\right) dV \\ =& \frac{1}{2}4\pi \int x^2 \left[\left(h^*\right)^2 + \left(g^*\right)^2\right] dx = \left(\frac{1}{2}\right) 4\pi I_2\end{aligned} \tag{80}$$

where $\hbar = 1$ and $I_2 \approx 0.063$. Most interesting here is that ψ is not a wave function of a quantum particle but it is the quark field which interacts in a nonlinear way with the scalar fields ϕ^A (which after above-mentioned simplifications present gluon fields). Consequently, ψ cannot be normalized to unity, i.e.

$$\int \left(h^2 + g^2\right) dV \neq 1. \tag{81}$$

This leads to a very interesting result: the z-projection of the total angular momentum of the solution is $\mathcal{M}_z < \hbar/2$[1]. This can be related to the spin problem of nucleon which raises experimental and theoretical questions regarding the contribution of the orbital momentum of the quarks to the total spin of the nucleon. One can suppose [4] that the full spin of nucleon has two components: one from gluon fields and another from quark field. Our calculations are in

[1] Note that for $g(0) = 0.2$ we have $\mathcal{M}_z > \hbar/2$.

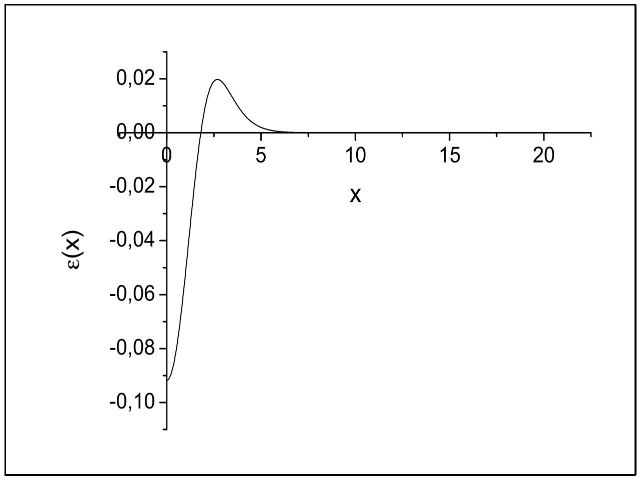

Figure 9: The profile of the energy density.

agreement with this statement: the angular momentum from the quark field is non-zero and can be $< \hbar/2$. Unfortunately, for the presented model we cannot calculate the term coming from the gluon fields since the gluon fields are presented here by the scalar field which does not have orbital momentum or spin. But it can be the aim for future investigations. For the simple model presented here, one can try to tune free parameters of the model, $(\lambda_{1,2}, M, \alpha, g(0))$ in such a way so as $\mathcal{M}_z = \hbar/2$, i.e. $\int \left(h^2 + g^2\right) dV = 1$, is fulfilled.

6 Discussion, conclusion and problems

In this paper, we have presented a model of nucleon based on the non-perturbative Heisenberg's quantization technique applied for the SU(3) gauge Yang-Mills theory plus quark field. We have shown that under some assumptions and simplifications one can reduce the gluon-quark Lagrangian to a scalar-fermion Lagrangian. The derived spherically symmetric solution of corresponding equations can be considered as a very simplified model of nucleon. Such a "nucleon" has a finite mass and spin $< \hbar/2$.

The comparison between the "nucleon" and the scalar glueball (which is "nucleon" minus quark field) shows us that there is *a mass gap* which is relatedyo the presence of the energy level E^* of the quark field. This value is $E^* \neq 0$ because it is the *eigenvalue* for the Dirac equation.

Another interesting feature of the "nucleon" is that $\mathcal{M}_z < \hbar/2$. One can hope that after some complication of the presented model an additional term from the gluon fields will appear and the sum of these two terms gives us the spin $= \hbar/2$. This could resolve the spin problem of the nucleon.

Now we would like to list the assumptions and simplifications which are nec-

essary for the reduction from the gluon-quark Lagrangian to the scalar-fermion Lagrangian:

- The infinite set of equations connecting Green's functions are reduced using the averaged Lagrangian.
- This reduction is based on the one-function approximation for the 2- and 4-point Green's functions.
- It is assumed that the quark field approximately is classical one.
- It is assumed that the interaction between gluon and quark fields is determined by their mean-square fluctuations: $\mathcal{L}_{int} = \left\langle \left(\widehat{\bar{\psi}}\widehat{\psi}\right) \left(\widehat{A}^B_\mu \widehat{A}^B_\mu\right) \right\rangle$.

Also, we would like to mention the following problems arising in the context of the considered model:

- It is necessary to prove the existence of the presented regular solution. Mathematically, this means that one should give an exact ground of the presented method of the numerical solution of the non-linear eigenvalue problem (21)-(24).
- From the physical point of view, it is necessary to generalize this model so that it includes the gluon orbital momentum which can give a contribution to spins of the glueball and the nucleon.
- To compare the presented value of the energy with the mass of nucleon it is necessary to tune up free parameters of the model. It is not a simple procedure since the preliminary numerical analysis shows us that the regular solutions exist not for all values of the parameters.
- It is interesting to investigate the other possible variants of the interaction between gluon and quark fields. In this connection, one can mention Ref. [6], in which a model of nucleon is derived on the basis of a "chromo dielectric model". One can suppose that there exists such a gluon-quark interaction that will lead to the "chromo dielectric model".

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References

[1] V. Dzhunushaliev, "Scalar model of the glueball", hep-ph/0312289, to appear in Found. Phys. Lett.

[2] W. Heisenberg, *Introduction to the unified field theory of elementary particles.*, Max - Planck - Institut für Physik und Astrophysik, Interscience Publishers London, New York, Sydney, 1966; W. Heisenberg, Nachr. Akad. Wiss. Göttingen N8, 111 (1953); W. Heisenberg, Zs. Naturforsch. **9a**, 292 (1954); W. Heisenberg, F. Kortel und H. Mütter, Zs. Naturforsch. **10a**, 425 (1955); W. Heisenberg, Zs. für Phys. **144**, 1 (1956); P. Askali and W. Heisenberg, Zs. Naturforsch. **12a**, 177 (1957); W. Heisenberg, Nucl. Phys. **4**, 532 (1957); W. Heisenberg, Rev. Mod. Phys. **29**, 269 (1957).

[3] V. Dzhunushaliev and D. Singleton, Mod. Phys. Lett. **A18**, 2873 (2003).

[4] D. Singleton, Mod. Phys. Lett. **A16**, 41 (2001).

[5] G.H. Derrick, J. Math. Phys. **5**, 1252 (1964).

[6] G. Martens, C. Greiner, S. Leupold, and U. Mosel, "Two- and three-body color flux tubes in the Chromo Dielectric Model", hep-ph/0407215.